PENETRABLE WEDGE ANALYSIS

Robert W. Scharstein
Department of Electrical Engineering

and

Anthony M. J. Davis
Department of Mathematics

The University of Alabama
Tuscaloosa, AL 35487

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PREFACE

Two complementary analyses of the time-harmonic scattering by a penetrable wedge are presented. The distance from the apex (appropriately scaled by the wavenumber in the exterior region) of the exciting line source is the single length scale in this infinite-domain boundary value problem. The work summarized herein represents two mathematical approaches (among a series of candidates) to solve this important scattering problem and to visualize the wave physics.

ACOUSTICAL SCATTERING BY A PENETRABLE WEDGE

Anthony M.J. Davis

Mathematics Department
University of Alabama
Tuscaloosa, AL 35487-0350

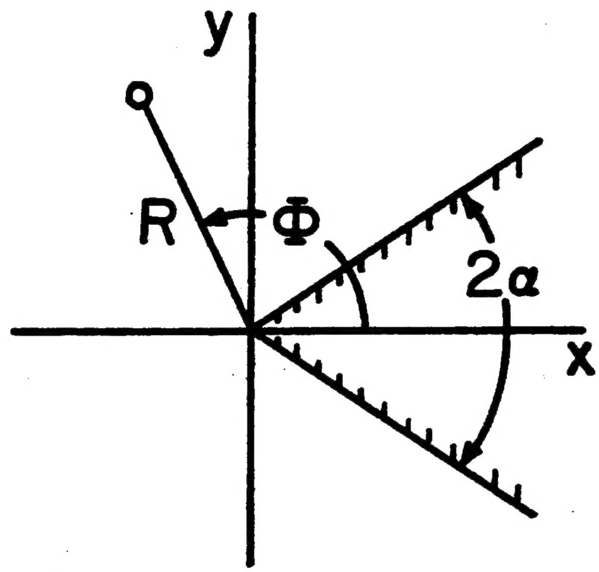

Figure 1: Wedge Geometry

1 Introduction

Consider the two-dimensional scattering of a time-harmonic sound wave generated by a line source and incident upon a penetrable wedge of angle 2α which, to simplify the presentation, is assumed in the main text to be such that π/α is an integer m (Figure 1). The wave speeds in the interior and exterior of the wedge are distinct and the radiation condition of only outgoing waves at infinity is applied in all directions. At the boundary of the wedge there is a pair of transmission conditions which ensure continuity of the acoustic pressure and normal velocity. Additional simplification is achieved by assuming that only one side of the wedge is directly 'lit' by the source. Such a transmission problem can be formulated, by use of

the free space Green's function in each region, in terms of a pair of coupled integral equations for two unknowns, the pressure and normal velocity, over the total boundary of the wedge. The details are given by Kleinman and Martin (1988) for the finite transmitting body.

However, by using suitably modified Green's functions and considering separately the symmetric and antisymmetric parts of the pressure field with respect to the center plane of the wedge, a pair of disjoint integral equations of the first kind can be obtained for the two parts of the normal velocity on just one face of the wedge. Transformation to equations of the second kind is then achieved by using a technique for solving integral equations with Hankel function kernels (Porter, 1983). The procedure described here can be regarded as a perturbation from the case of the impenetrable hard wedge and this context is adopted for the sake of extra simplification of the presentation.

2 Formulation of the Transmission Problem

A sound wave generated by a z-directed source of period $2\pi/\omega$ is incident on a penetrable wedge which, in terms of cylindrical polar coordinates (r, ϕ, z), occupies the region $r > 0, -\alpha < \phi < \alpha$ (Figure 1). With the time factor $e^{i\omega t}$ suppressed, the induced exterior and interior pressure fields, u_e and u_i respectively, satisfy the wave equations

$$(\nabla^2 + k_e^2)u_e = 0, \qquad (|\phi| > \alpha), \tag{1}$$

$$(\nabla^2 + k_i^2)u_i = 0, \qquad (|\phi| < \alpha), \tag{2}$$

while the incident field, with the source at (R, Φ), is given by

$$u_{inc} = \frac{1}{4} i H_0^{(2)}[k_e(r^2 + R^2 - 2rR\cos(\phi - \Phi))^{1/2}], \tag{3}$$

where $\alpha < \Phi < \pi - \alpha$, in accordance with the assumption that only one side of the wedge is directly 'lit' by the source. The transmission conditions at the interfaces are

$$u = u_i, \qquad \frac{\partial u}{\partial \phi} = \rho \frac{\partial u_i}{\partial \phi} \qquad (\phi = \pm\alpha), \tag{4}$$

where

$$u = u_e + u_{inc} \tag{5}$$

denotes the total pressure field in the exterior region. At infinity, all fields contain only outgoing waves. The wavenumbers k_e, k_i and the density

ratio ρ are given real, positive constants with $k_i > k_e$. Now introduce the symmetric and antisymmetric parts of u_{inc}, u_e, u_i by writing, for each field

$$u^{(\pm)}(r,\phi) = \frac{1}{2}[u(r,\phi) \pm u(r,-\phi)] \tag{6}$$

and, similarly, set

$$f^{(\pm)}(r) = \frac{1}{r}\frac{\partial u_i^{(\pm)}}{\partial \phi}(r,\alpha) \qquad (r > 0). \tag{7}$$

Then the exterior and interior regions are reduced to $\alpha < \phi < \pi$ and $0 < \phi < \alpha$ respectively, with the differential equations (1) and (2) satisfied therein and the transmission conditions (4) applicable at the sole interface $\phi = \alpha$. The additional hard and soft conditions are

$$\begin{aligned} &\frac{\partial u_e^{(+)}}{\partial \phi} = 0, \qquad u_e^{(-)} = 0 \qquad (\phi = \pi), \\ &\frac{\partial u_i^{(+)}}{\partial \phi} = 0, \qquad u_i^{(-)} = 0 \qquad (\phi = 0). \end{aligned} \tag{8}$$

Let $G_e^{(\pm)}(r,\phi;r',\phi')$ $(\alpha < \phi, \phi' < \pi)$ be two Green's functions defined in the exterior region where they satisfy (1) except at the singularity in the prescribed term $\frac{1}{4}iH_0^{(2)}[k_e(r^2 + r'^2 - 2rr'\cos(\phi - \phi'))^{1/2}]$. Similarly, let $G_i^{(\pm)}(r,\phi;r',\phi')$ $(0 < \phi, \phi' < \alpha)$ be two Green's functions defined in the interior region where they satisfy (2) except at the singularity in the prescribed term $\frac{1}{4}iH_0^{(2)}[k_i(r^2 + r'^2 - 2rr'\cos(\phi - \phi'))^{1/2}]$. These four functions satisfy the respective hard and soft conditions (8) and all of them are required to have zero normal derivative at the interface $\phi = \alpha$ and only outgoing waves at infinity.

Then when Green's theorem is applied to $u_e^{(\pm)}(r,\phi) - [u_e^{(\pm)}(r,\phi)]_{\rho=0}$ and $G_e^{(\pm)}(r,\phi;r',\alpha)$ in the external region $\alpha < \phi < \pi$ and to $u_i^{(\pm)}(r,\phi)$ and $G_i^{(\pm)}(r,\phi;r',\alpha)$ in the interior region $0 < \phi < \alpha$, the resulting pairs of integral equations on the wedge boundary are

$$u_e^{(\pm)}(r,\alpha) - [u_e^{(\pm)}(r,\alpha)]_{\rho=0} = \frac{1}{\pi}\int_0^\infty G_e^{(\pm)}(r,\alpha;r',\alpha)\frac{1}{r'}\frac{\partial u^{(\pm)}}{\partial \phi'}(r',\alpha)\,dr'$$

$$= \frac{\rho}{\pi}\int_0^\infty G_e^{(\pm)}(r,\alpha;r',\alpha) f^{(\pm)}(r')dr' \qquad (r > 0)$$

and

$$u_i^{(\pm)}(r,\alpha) = -\frac{1}{\pi}\int_0^\infty G_i^{(\pm)}(r,\alpha;r',\alpha)\frac{1}{r'}\frac{\partial u_i^{(\pm)}}{\partial \phi'}(r',\alpha)\,dr'$$

$$= -\frac{1}{\pi}\int_0^\infty G_i^{(\pm)}(r,\alpha;r',\alpha) f^{(\pm)}(r')dr' \qquad (r > 0),$$

after use of (4), (5) and (7). Now (4) and (5) allow the unknown functions on the left hand sides to be eliminated to obtain a pair of disjoint integral equations of the first kind for the symmetric and antisymmetric source density functions $f^{(\pm)}$, namely

$$u_{inc}^{(\pm)}(r,\alpha) + [u_e^{(\pm)}(r,\alpha)]_{\rho=0} = [u^{(\pm)}(r,\alpha)]_{\rho=0}$$

$$= -\frac{1}{\pi}\int_0^\infty \left[G_i^{(\pm)}(r,\alpha;r',\alpha) + \rho G_e^{(\pm)}(r,\alpha;r',\alpha)\right] f^{(\pm)}(r')dr' \qquad (r > 0). \tag{9}$$

This is the equation whose reduction to a suitable integral equation of the second kind is the purpose of this work. Note that $\rho = 0$ for the hard wedge, in which limit f becomes irrelevant to the scattering problem because no transmission occurs in this case. However, the structure of the solution

procedure presented below may be regarded as a perturbation about this limit. On writing

$$f^{(\pm)} = f_0^{(\pm)} + \rho f_1^{(\pm)} + \rho^2 f_2^{(\pm)} + ..., \tag{10}$$

the integral equation (9) reduces to the system

$$u_0^{(\pm)}(r) = [u^{(\pm)}(r,\alpha)]_{\rho=0} = -\frac{1}{\pi}\int_0^\infty G_i^{(\pm)}(r,\alpha;r',\alpha) f_0^{(\pm)}(r')dr',$$

$$\frac{1}{\pi}\int_0^\infty G_e^{(\pm)}(r,\alpha;r',\alpha) f_{n-1}^{(\pm)}(r')dr' = -\frac{1}{\pi}\int_0^\infty G_i^{(\pm)}(r,\alpha;r',\alpha) f_n^{(\pm)}(r')dr' \quad (n \geq 1)$$

$$(r > 0). \tag{11}$$

Thus each term in the expansion (10) is determined from an interior problem while each forcing term, after the first, arises from an exterior problem. In particular, this shows that the Green's function $G_i^{(\pm)}$, with wavenumber k_i plays the dominant role in the kernel of (9).

3 The Green's Functions and Forcing Term

With the incident field defined by (3), it may be deduced from Jones (1986, section 9.19) that the forcing term in (9) is given by

$$u_0^{(\pm)}(r) = [u^{(\pm)}(r,\alpha)]_{\rho=0} = \frac{1}{16(\pi-\alpha)}\int_{\infty-\pi i}^{\infty+\pi i} H_0^{(2)}[k_e(r^2+R^2-2rR\cosh\tau)^{1/2}]$$

$$\times \sinh\frac{\pi\tau}{2(\pi-\alpha)}\left[\frac{1}{\cosh\frac{\pi\tau}{2(\pi-\alpha)} - \cos\frac{\pi(\Phi-\alpha)}{2(\pi-\alpha)}} \pm \frac{1}{\cosh\frac{\pi\tau}{2(\pi-\alpha)} + \cos\frac{\pi(\Phi+\alpha)}{2(\pi-\alpha)}}\right] d\tau$$

$$= \frac{1}{4} i H_0^{(2)}[k_e(r^2 + R^2 - 2rR\cos(\Phi-\alpha))^{1/2}]$$

$$+\frac{1}{16(\pi-\alpha)}\int_{-\infty}^{\infty} H_0^{(2)}[k_e(r^2+R^2+2rR\cosh u)^{1/2}]\sinh\frac{\pi(u+\pi i)}{2(\pi-\alpha)}$$

$$\times\left[\frac{1}{\cosh\frac{\pi(u+\pi i)}{2(\pi-\alpha)}-\cos\frac{\pi(\Phi-\alpha)}{2(\pi-\alpha)}}\pm\frac{1}{\cosh\frac{\pi(u+\pi i)}{2(\pi-\alpha)}+\cos\frac{\pi(\Phi+\alpha)}{2(\pi-\alpha)}}\right]du, \qquad (12)$$

in which only the first term contributes to the residue at $\tau = 0$ because of the assumption $\Phi + \alpha < \pi$.

Similarly the external Green's functions are given by

$$G_e^{(+)}(r,\phi;r',\alpha)=\frac{1}{8(\pi-\alpha)}\int_{\infty-\pi i}^{\infty+\pi i}\frac{H_0^{(2)}[k_e(r^2+r'^2-2rr'\cosh\tau)^{1/2}]\sinh\frac{\pi\tau}{\pi-\alpha}}{\cosh\frac{\pi\tau}{\pi-\alpha}+\cos\frac{\pi(\pi-\phi)}{\pi-\alpha}}d\tau,$$

i.e.

$$G_e^{(+)}(r,\alpha;r',\alpha)=\frac{1}{4}iH_0^{(2)}[k_e|r-r'|]$$

$$+\frac{1}{8(\pi-\alpha)}\int_{-\infty}^{\infty}\frac{H_0^{(2)}[k_e(r^2+r'^2+2rr'\cosh u)^{1/2}]}{\tanh\frac{\pi(u+\pi i)}{2(\pi-\alpha)}}du \qquad (13)$$

and

$$G_e^{(-)}(r,\phi;r',\alpha)=\frac{\sin\frac{\pi(\pi-\phi)}{2(\pi-\alpha)}}{4(\pi-\alpha)}\int_{\infty-\pi i}^{\infty+\pi i}\frac{H_0^{(2)}[k_e(r^2+r'^2-2rr'\cosh\tau)^{1/2}]\sinh\frac{\pi\tau}{2(\pi-\alpha)}}{\cosh\frac{\pi\tau}{\pi-\alpha}+\cos\frac{\pi(\pi-\phi)}{\pi-\alpha}}d\tau$$

i.e.

$$G_e^{(-)}(r,\alpha;r',\alpha)=\frac{1}{4}iH_0^{(2)}[k_e|r-r'|]$$

$$+\frac{1}{8(\pi-\alpha)}\int_{-\infty}^{\infty}\frac{H_0^{(2)}[k_e(r^2+r'^2+2rr'\cosh u)^{1/2}]}{\sinh\frac{\pi(u+\pi i)}{2(\pi-\alpha)}}du. \qquad (14)$$

The assumption that $\alpha = \pi/2m$, for some integer $m > 1$, simplifies the corresponding expressions for the interior Green's functions which evidently can be constructed by adding image sources and sinks to the prescribed

source. Thus

$$G_i^{(\pm)}(r,\phi;r',\alpha) = \frac{1}{4}i\sum_{n=1}^{n=m}(\pm 1)^n \left\{H_0^{(2)}[k_i(r^2+r'^2-2rr'\cos[(2n-1)\alpha+\phi])^{1/2}]\right.$$

$$\left.\pm H_0^{(2)}[k_i(r^2+r'^2-2rr'\cos[(2n-1)\alpha-\phi])^{1/2}]\right\}$$

and, in particular,

$$G_i^{(\pm)}(r,\alpha;r',\alpha) = \frac{1}{4}i\left\{H_0^{(2)}[k_i|r-r'|] + (\pm 1)^m H_0^{(2)}[k_i(r+r')]\right.$$

$$\left. +2\sum_{n=1}^{n=m-1}(\pm 1)^n H_0^{(2)}[k_i(r^2+r'^2-2rr'\cos n\pi/m)^{1/2}]\right\}. \qquad (15)$$

4 Inversion of the Hankel Kernel

Consider the integral equation of the first kind

$$g(r) = \int_0^\infty \psi(r')H_0^{(2)}(k_i|r-r'|)dr' \qquad (r>0), \qquad (16)$$

whose solution, which may have an integrable singularity at $x = 0$, is assumed to be bounded at infinity, and define the operator L_i by

$$\begin{aligned}(L_i\psi)(s) &= -\psi'(s) + k_i\int_s^\infty \psi(r)\frac{J_1[k_i(r-s)]}{r-s}dr \\ &= \left(\frac{d^2}{ds^2}+k_i^2\right)\int_s^\infty \psi(r)J_0[k_i(r-s)]dr \qquad (s\geq 0). \qquad (17)\end{aligned}$$

Porter (1983) showed that application of this operator to (16) yields the singular integral equation

$$(L_ig)(s) = -\frac{2i}{\pi}\int_0^\infty \psi(r')\frac{e^{ik_i(r'-s)}}{r'-s}dr' \qquad (s\geq 0), \qquad (18)$$

whose solution is given by Muskhelishvili (1953). The complementary function, $e^{-ik_i r'}/\sqrt{r'}$, satisfies the outgoing wave condition and furnishes a unique solution bounded at the origin, namely

$$\psi(r) = -\frac{i}{2\pi}\sqrt{r}\int_0^\infty \frac{(L_i g)(s)}{\sqrt{s}}\frac{e^{ik_i(s-r)}}{s-r}ds \qquad (r \geq 0). \tag{19}$$

Thus, on writing (15) in the form

$$G_i^{(\pm)}(r,\alpha;r',\alpha) = \frac{1}{4}i\left[H_0^{(2)}(k_i|r-r'|) + K_i^{(\pm)}(r,r')\right], \tag{20}$$

the first of the integral equations (11) can be rearranged as

$$4\pi i u_0^{(\pm)}(r) - \int_0^\infty K_i^{(\pm)}(r,r')f_0^{(\pm)}(r')dr' = \int_0^\infty H_0^{(2)}(k_i|r-r'|)f_0^{(\pm)}(r')dr',$$

which is of the type (16) when the left hand side is regarded as known. Hence, from (19),

$$f_0^{(\pm)}(r) = \sqrt{r}\int_0^\infty \frac{e^{ik_i(s-r)}}{\sqrt{s}(s-r)}\left[2(L_i u_0^{(\pm)})(s) + \frac{i}{2\pi}\int_0^\infty (L_i K_i^{(\pm)})(s,r')f_0^{(\pm)}(r')dr'\right]ds$$

$$(r \geq 0), \tag{21}$$

which is an integral equation of the second kind for each zero order density function. The two integrations introduced by the above inversion procedure can be evaluated exactly when plane wave representations are used.

The inversion of the remaining integral equations in (11) can be achieved similarly, except that the forcing terms also have a Hankel kernel, with wavenumber k_e, as given by (13) and (14) for the symmetric and antisymmetric cases. On writing these equations in the form

$$G_e^{(\pm)}(r,\alpha;r',\alpha) = \frac{1}{4}i\left[H_0^{(2)}(k_e|r-r'|) + K_e^{(\pm)}(r,r')\right], \tag{22}$$

as in (20) for the interior Green's functions, the sequence of integral equations in (11) can be arranged as

$$
\begin{aligned}
&-\int_0^\infty \left[K_e^{(\pm)}(r,r')f_{n-1}^{(\pm)}(r') + K_i^{(\pm)}(r,r')f_n^{(\pm)}(r')\right] dr' \\
&-\int_0^\infty \left[H_0^{(2)}(k_e|r-r'|) - H_0^{(2)}(k_i|r-r'|)\right] f_{n-1}^{(\pm)}(r')dr' \\
&\quad = \int_0^\infty H_0^{(2)}(k_i|r-r'|)\left[f_n^{(\pm)}(r') + f_{n-1}^{(\pm)}(r')\right] dr',
\end{aligned}
$$

which also can be regarded as of the type (16). Hence, on defining the bounded kernel

$$
K_s(r,r') = H_0^{(2)}(k_e|r-r'|) - H_0^{(2)}(k_i|r-r'|) \qquad (r,r' \geq 0), \tag{23}
$$

the inversion formula (19) yields

$$
\begin{aligned}
f_n^{(\pm)}(r) + f_{n-1}^{(\pm)}(r) = \frac{i\sqrt{r}}{2\pi}\int_0^\infty \frac{e^{ik_i(s-r)}}{\sqrt{s(s-r)}}\int_0^\infty \Big[(L_iK_e^{(\pm)})(s,r')f_{n-1}^{(\pm)}(r') \\
+(L_iK_s)(s,r')f_{n-1}^{(\pm)}(r') + (L_iK_i^{(\pm)})(s,r')f_n^{(\pm)}(r')\Big] dr'ds \qquad (n \geq 1) \qquad (r \geq 0),
\end{aligned} \tag{24}
$$

which is an integral equation of the second kind for $f_n^{(\pm)} (n \geq 1)$ when $f_{n-1}^{(\pm)}$ is already determined.

The expansion (10) enables equations (21) and (24) to be combined to show that the corresponding inversion of (9) is

$$
\begin{aligned}
(1+\rho)f^{(\pm)}(r) = \sqrt{r}\int_0^\infty \frac{e^{ik_i(s-r)}}{\sqrt{s(s-r)}}\Big[2(L_iu_0^{(\pm)})(s) + \frac{i}{2\pi}\int_0^\infty \Big\{(L_iK_i^{(\pm)})(s,r') \\
+\rho(L_iK_e^{(\pm)})(s,r') + \rho(L_iK_s)(s,r')\Big\} f^{(\pm)}(r')dr'\Big] ds \qquad (r \geq 0),
\end{aligned} \tag{25}
$$

which is an integral equation of the second kind for $f^{(\pm)}$.

5 Evaluation of the Inversion Integrals

The two integrations introduced by the above inversion procedure can be evaluated exactly when plane wave representations are used. Consider the inversion of $e^{-i\beta r}$ for real values of β. First, it is readily shown from the definition (17) that

$$(L_i e^{-i\beta r})(s) = e^{-i\beta s} \begin{cases} (k_i^2 - \beta^2)^{1/2} & (|\beta| < k_i) \\ i\operatorname{sgn}\beta(\beta^2 - k_i^2)^{1/2} & (|\beta| > k_i) \end{cases} \tag{26}$$

Second, the evaluation of

$$\sqrt{r}\int_0^\infty \frac{e^{ik_i(s-r)}}{\sqrt{s(s-r)}} e^{-i\beta s} ds$$

requires that of the integral I given by

$$I(\lambda, r) = \sqrt{r}\int_0^\infty \frac{e^{i\lambda(s-r)}}{\sqrt{s(s-r)}} ds \qquad (r > 0), \tag{27}$$

which vanishes at $\lambda = 0$ and, by consideration of $\frac{\partial I}{\partial \lambda}$, can be expressed in terms of the Fresnel integrals C_1 and S_1 defined by

$$C_1(x) + iS_1(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{iy^2} dy. \tag{28}$$

Thus

$$I = -\pi\sqrt{2}e^{-i\pi/4}\left\{C_1[(\lambda r)^{1/2}] - iS_1[(\lambda r)^{1/2}]\right\} \qquad (\lambda > 0),$$

$$I = -\pi\sqrt{2}e^{i\pi/4}\left\{C_1[(|\lambda| r)^{1/2}] + iS_1[(|\lambda| r)^{1/2}]\right\} \qquad (\lambda < 0). \tag{29}$$

Hence it follows from (26) and (29) that

$$\sqrt{r}\int_0^\infty \frac{e^{ik_i(s-r)}}{\sqrt{s(s-r)}}(L_i e^{-i\beta r})(s) ds = \pi\sqrt{2}e^{-i\beta r} M_i(\beta, r), \tag{30}$$

where

$$M_i(\beta, r) = \begin{cases} (\beta^2 - k_i^2)^{1/2} e^{-i\pi/4} \left\{ C_1([(\beta - k_i)r]^{1/2}) + iS_1([(\beta - k_i)r]^{1/2}) \right\} & (\beta > k_i) \\ -(k_i^2 - \beta^2)^{1/2} e^{-i\pi/4} \left\{ C_1([(k_i - \beta)r]^{1/2}) - iS_1([(k_i - \beta)r]^{1/2}) \right\} & (|\beta| < k_i) \\ (\beta^2 - k_i^2)^{1/2} e^{i\pi/4} \left\{ C_1([(k_i - \beta)r]^{1/2}) - iS_1([(k_i - \beta)r]^{1/2}) \right\} & (\beta < -k_i) \end{cases} \tag{31}$$

Now in order to use (30), it is necessary to use plane wave representations of the functions to be inverted on the right hand side of (25). The standard representations of J_0 and Y_0 (Abramovitz and Stegun, 1964) are combined in the integral representation

$$H_0^{(2)}(k|r - r'|) = \frac{1}{\pi} \int_C e^{-ik(r-r')\cos\tau} d\tau, \tag{32}$$

where C is the contour from $-i\infty$ to $\pi + i\infty$ drawn along the negative imaginary axis, the real axis from 0 to π and a line parallel to the imaginary axis. Equation (32), which allows (18) to be derived from (16), can be generalized to

$$H_0^{(2)}[k\{(x - x')^2 + (y - y')^2\}^{1/2}] = \frac{1}{\pi} \int_C e^{-ik[(x-x')\cos\tau + |y-y'|\sin\tau]} d\tau. \tag{33}$$

Similarly, the integral representation

$$H_0^{(2)}(k|r - r'|) = -\frac{1}{\pi i} \int_{-\infty}^{\infty} e^{-ik|r-r'|\cosh v} dv$$

can be generalized to

$$H_0^{(2)}[k\{(x - x')^2 - (y - y')^2\}^{1/2}] = -\frac{1}{\pi i} \int_{-\infty}^{\infty} e^{-ik[|x-x'|\cosh v + (y-y')\sinh v]} dv$$

$$(|x - x'| > y - y'). \tag{34}$$

Now the forcing and kernel functions in (25) can be reduced, as follows, to single or rapidly convergent double integrals involving at most the Fresnel integrals. The function $u_0^{(\pm)}(r)$, given by (12), can be written in the form

$$\begin{aligned} u_0^{(\pm)}(r) &= \frac{1}{4} i H_0^{(2)}[k_e(r^2+R^2-2rR\cos(\Phi-\alpha))^{1/2}] \\ &+ \frac{1}{2}\int_{-\infty}^{\infty} H_0^{(2)}[k_e(r^2+R^2+2rR\cosh u)^{1/2}] h^{(\pm)}(u)du \\ &= \frac{i}{4\pi}\int_C e^{-ik_e[r\cos\tau - R\cos(\Phi-\alpha+\tau)]}d\tau \\ &- \frac{1}{2\pi i}\int_{-\infty}^{\infty} h^{(\pm)}(u)du \int_{-\infty}^{\infty} e^{-ik_e[r\cosh v + R\cosh(u+v)]}dv, \qquad (35) \end{aligned}$$

by use of (33) and (34), and hence, from (30),

$$2\sqrt{r}\int_0^{\infty} \frac{e^{ik_i(s-r)}}{\sqrt{s(s-r)}}(L_i u_0^{(\pm)})(s)ds = \frac{i}{\sqrt{2}}\int_C M_i(k_e\cos\tau, r)e^{-ik_e[r\cos\tau - R\cos(\Phi-\alpha+\tau)]}d\tau$$

$$+ i\sqrt{2}\int_{-\infty}^{\infty} h^{(\pm)}(u)du \int_{-\infty}^{\infty} M_i(k_e\cosh v, r)e^{-ik_e[r\cosh v + R\cosh(u+v)]}dv. \qquad (36)$$

where M_i is defined by (31). The contribution from the interior Green's function arises, according to (15) and (20), from

$$K_i^{(\pm)}(r,r') = (\pm 1)^m H_0^{(2)}[k_i(r+r')] + 2\sum_{n=1}^{n=m-1}(\pm 1)^n H_0^{(2)}[k_i(r^2+r'^2-2rr'\cos n\pi/m)^{1/2}]$$

$$= (\pm 1)^m H_0^{(2)}[k_i(r+r')] + 2\int_C e^{-ik_i r\cos\tau}\sum_{n=1}^{n=m-1}(\pm 1)^n e^{-ik_i r'\cos(n\pi/m+\tau)}d\tau,$$

by use of (33). The first term can be inverted by comparison with (18) and hence, after further use of (30),

$$\frac{i\sqrt{r}}{2\pi}\int_0^{\infty} \frac{e^{ik_i(s-r)}}{\sqrt{s(s-r)}}(L_i K_i^{(\pm)})(s,r')ds = \frac{(\pm 1)^m}{\pi}\left(\frac{r}{r'}\right)\frac{e^{ik_i(r+r')}}{r+r'}$$

$$+ i\sqrt{2}\int_C e^{-ik_i r\cos\tau} M_i(k_i\cos\tau, r) \sum_{n=1}^{n=m-1} (\pm 1)^n e^{-ik_i r'\cos(n\pi/m+\tau)} d\tau. \qquad (37)$$

The contribution from the exterior Green's function arises from $K_e^{\pm}$, given by (13), (14) and (22), and K_s, defined by (23). On using (34) and then (30), it follows that

$$\frac{i\sqrt{r}}{2\pi}\int_0^\infty \frac{e^{ik_i(s-r)}}{\sqrt{s(s-r)}}(L_i K_e^{(+)})(s,r')ds = \frac{i}{2\sqrt{2}\pi(\pi-\alpha)}\times$$

$$\int_{-\infty}^{\infty} \frac{du}{\tanh\frac{\pi(u+\pi i)}{2(\pi-\alpha)}} \int_{-\infty}^{\infty} M_i(k_e\cosh v, r) e^{-ik_e[r\cosh v + r'\cosh(u+v)]} dv. \qquad (38)$$

Similarly

$$\frac{i\sqrt{r}}{2\pi}\int_0^\infty \frac{e^{ik_i(s-r)}}{\sqrt{s(s-r)}}(L_i K_e^{(-)})(s,r')ds = \frac{i}{2\sqrt{2}\pi(\pi-\alpha)}\times$$

$$\int_{-\infty}^{\infty} \frac{du}{\sinh\frac{\pi(u+\pi i)}{2(\pi-\alpha)}} \int_{-\infty}^{\infty} M_i(k_e\cosh v, r) e^{-ik_e[r\cosh v + r'\cosh(u+v)]} dv. \qquad (39)$$

For the remaining term in (25), write

$$\begin{aligned}(L_i K_s)(s,r') &= [L_e H_0^{(2)}(k_e|r-r'|)](s,r') - [L_i H_0^{(2)}(k_i|r-r'|)](s,r') \\ &\quad -[(L_e - L_i)H_0^{(2)}(k_e|r-r'|)](s,r') \\ &= -\frac{2i}{\pi}\frac{e^{ik_e(r'-s)} - e^{ik_i(r'-s)}}{r'-s} \\ &\quad - [(L_e - L_i)H_0^{(2)}(k_e|r-r'|)](s,r') \qquad (s, r' \geq 0), \qquad (40)\end{aligned}$$

as in the derivation of (18). The contribution of the first term to

$$\frac{i\sqrt{r}}{2\pi}\int_0^\infty \frac{e^{ik_i(s-r)}}{\sqrt{s(s-r)}}(L_i K_s)(s,r')ds$$

is therefore

$$\frac{\sqrt{r}}{\pi^2}e^{ik_i(r'-r)}\int_0^\infty \frac{e^{i(k_i-k_e)(s-r)}-1}{\sqrt{s(s-r)(r'-s)}}ds$$

$$=\frac{1}{\pi^2}\frac{e^{ik_i(r'-r)}}{r'-r}[I(k_i-k_e,r)-\left(\frac{r}{r'}\right)^{1/2}I(k_i-k_e,r')],$$

where I is defined by (27). For the second term in (40), rewrite (32) as

$$H_0^{(2)}(k|r-r'|)](s,r')=\frac{1}{\pi}\int_{-1}^1 e^{-ik(r-r')v}\frac{dv}{(1-v^2)^{1/2}}$$

$$+\frac{i}{\pi}\int_1^\infty[e^{-ik(r-r')v}+e^{ik(r-r')v}]\frac{dv}{(v^2-1)^{1/2}} \tag{41}$$

and observe that the formula for (L_e-L_i) corresponding to (26) is

$$[(L_e-L_i)e^{-ik_evr}](s)=e^{-ik_evs}\begin{cases} k_e(1-v^2)^{1/2}-(k_i^2-k_e^2v^2)^{1/2} & (|v|<1)\\ i\operatorname{sgn}v k_e(v^2-1)^{1/2}-(k_i^2-k_e^2v^2)^{1/2} & (1<|v|<k_i/k_e)\\ i\operatorname{sgn}v[k_e(v^2-1)^{1/2}-(k_e^2v^2-k_i^2)^{1/2}] & (|v|>k_i/k_e)\end{cases} \tag{42}$$

The factors on the right hand side of (42) effectively replace those in (26) and account must be taken of this when adapting (30) in order to use the function M_i given by (31). Hence the above results imply that

$$\frac{i\sqrt{r}}{2\pi}\int_0^\infty \frac{e^{ik_i(s-r)}}{\sqrt{s(s-r)}}(L_iK_s)(s,r')ds=\frac{1}{\pi^2}\frac{e^{ik_i(r'-r)}}{r'-r}[I(k_i-k_e,r)-\left(\frac{r}{r'}\right)^{1/2}I(k_i-k_e,r')]$$

$$+\frac{i}{\pi\sqrt{2}}\int_{-1}^1 e^{-ik_e(r-r')v}M_i(k_ev,r)\left[\frac{k_e}{(k_i^2-k_e^2v^2)^{1/2}}-\frac{1}{(1-v^2)^{1/2}}\right]dv$$

$$-\frac{1}{\pi\sqrt{2}}\int_1^\infty[e^{-ik(r-r')v}M_i(k_ev,r)\psi(v)+e^{ik(r-r')v}M_i(-k_ev,r)\psi(-v)]dv, \tag{43}$$

where I is given by (29) and

$$\psi(v)=\begin{cases}\frac{i\operatorname{sgn}v k_e}{(k_i^2-k_e^2v^2)^{1/2}}-\frac{1}{(v^2-1)^{1/2}} & (1<|v|<k_i/k_e)\\ \frac{k_e}{(k_e^2v^2-k_i^2)^{1/2}}-\frac{1}{(v^2-1)^{1/2}} & (|v|>k_i/k_e)\end{cases} \tag{44}$$

Evidently the expression (43) is bounded at $r = r'$, with the integral remaining convergent in this limit because the integrand is $O(v^{-2})$ as $v \to \infty$.

Thus equations (36-39) and (43) furnish simplified forms of the integrals on the right side of the integral equation (25). Additional similar terms will be introduced on relaxation of either or both of the assumptions $\alpha = \pi/2m$ and $\Phi + \alpha < \pi$. Asymptotic estimates for large r show that $f^{(\pm)}$ must be a linear combination of $e^{-ik_i r}/\sqrt{r}$ and $e^{-ik_e r}/\sqrt{r}$ in this limit, as might be anticipated. This information will facilitate a numerical solution for the even and odd components of the normal velocities at the interfaces of the wedge of angle 2α.

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PENETRABLE WEDGE SCATTERING VIA A PAIR OF COUPLED INTEGRAL EQUATIONS OF THE SECOND KIND

Robert W. Scharstein, Associate Professor
Department of Electrical Engineering

The University of Alabama
Tuscaloosa, Alabama 35487-0286
205-348-1761

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Abstract. A pair of coupled integral equations of the second kind are specialized to the even and odd symmetry components for the scattering by the penetrable wedge. The unknowns are both the scalar field and its normal derivative on the two wedge surfaces that separate the interior and exterior regions of different wave speeds and constitutive parameters (density or dielectric constant, depending on the specific physical application). A discretization and collocation procedure transforms the operator equations into coupled matrix equations, which are solved numerically. Further extensions and plans to include anticipated asymptotic behavior far from the line source and wedge apex are outlined.

I. Adaptation of Kleinman and Martin Analysis

The transmission problem for the penetrable wedge of Figure 1 consists of appropriate exterior and interior waves

$$\left(\nabla^2 + k_e^2\right) u_e(r,\phi) = -\frac{1}{r}\delta(r-r_0)\delta(\phi-\phi_0) \qquad (\alpha \le \phi \le 2\pi - \alpha) \tag{1}$$

$$\left(\nabla^2 + k_i^2\right) u_i(r,\phi) = 0 \qquad (-\alpha \le \phi \le \alpha) \tag{2}$$

subject to the two-dimensional Sommerfeld radiation condition plus the continuity boundary conditions

$$u_e(r,\pm\alpha) = u_i(r,\pm\alpha) \tag{3}$$

$$\frac{\partial}{\partial\phi}u_e(r,\pm\alpha) = \rho\frac{\partial}{\partial\phi}u_i(r,\pm\alpha) \tag{4}$$

where $\rho = \rho_e/\rho_i$ is the ratio of exterior to interior constitutive parameters (ambient density or dielectric constant). A pair of integral equations derived by Kleinman and Martin [4] for the scalar field u and its normal derivative $\partial u/\partial n$ on the boundary of a penetrable obstacle, are adapted to the semi-infinite wedge faces of Figure 1 to effect an accurate numerical solution of the scattering problem.

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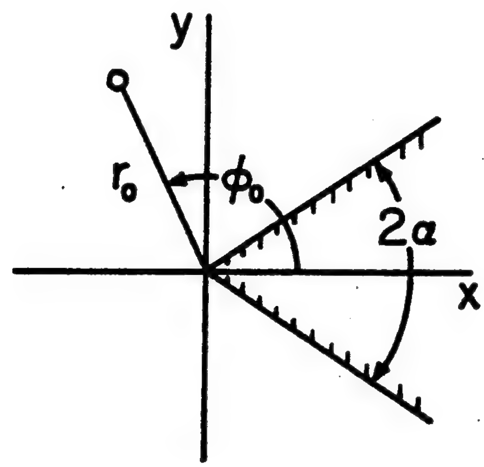

FIG.1. Penetrable Wedge and Line Source

Define the surface distributions

$$v_\pm(r) = u(r,\alpha) \pm u(r,-\alpha) \tag{5}$$

$$w_\pm(r) = \frac{\partial}{\partial n}u(r,\alpha) \pm \frac{\partial}{\partial n}u(r,-\alpha) \tag{6}$$

in order to succinctly decompose the field into its symmetric and antisymmetric components with respect to the wedge bisector (the x axis).

After careful algebra, the selected integral equations from [4] are written in a form defined on a single semi-infinite domain ($0 \le r < \infty$) to account for the even/odd symmetry:

$$\begin{aligned}(1+\rho)v_\pm(r) &\pm \int_0^\infty dr'\, v_\pm(r')\frac{\partial}{\partial n'}\left[G_e(r,\alpha;r',-\alpha) - \rho G_i(r,\alpha;r',-\alpha)\right] \\ &- \int_0^\infty dr'\, w_\pm(r')\left\{\left[G_e(r,\alpha;r',\alpha) - G_i(r,\alpha;r',\alpha)\right] \pm \left[G_e(r,\alpha;r',-\alpha) - G_i(r,\alpha;r',-\alpha)\right]\right\} \\ &= 2v_\pm^{\text{inc}}(r)\end{aligned} \tag{7}$$

$$\begin{aligned}(1+\rho)w_\pm(r) &\mp \int_0^\infty dr'\, w_\pm(r')\frac{\partial}{\partial n}\left[\rho G_e(r,\alpha;r',-\alpha) - G_i(r,\alpha;r',-\alpha)\right] + \rho \\ &\cdot \int_0^\infty dr'\, v_\pm(r')\frac{\partial^2}{\partial n \partial n'}\left\{\left[G_e(r,\alpha;r',\alpha) - G_i(r,\alpha;r',\alpha)\right] \pm \left[G_e(r,\alpha;r',-\alpha) - G_i(r,\alpha;r',-\alpha)\right]\right\} \\ &= 2\rho w_\pm^{\text{inc}}(r).\end{aligned} \tag{8}$$

This arrangement ensures regular or (at worst) weakly singular kernels, which involve the differences of the scaled interior and exterior free-space Green's functions and normal derivatives thereof:

$$\frac{\partial}{\partial n'}\left[G_e(r,\alpha;r',-\alpha) - \rho G_i(r,\alpha;r',-\alpha)\right] = \frac{\sin 2\alpha}{2i}\frac{r}{R}\left[\rho k_i H_1^{(1)}(k_i R) - k_e H_1^{(1)}(k_e R)\right] \quad (9)$$

$$\begin{aligned} G_e(r,\alpha;r',\alpha) - G_i(r,\alpha;r',\alpha) &= \frac{1}{2i}\left[H_0^{(1)}(k_e|r-r'|) - H_0^{(1)}(k_i|r-r'|)\right] \\ &\xrightarrow[r\to r']{} \frac{1}{\pi}\ln\frac{k_e}{k_i} \end{aligned} \quad (10)$$

$$G_e(r,\alpha;r',-\alpha) - G_i(r,\alpha;r',-\alpha) = \frac{1}{2i}\left[H_0^{(1)}(k_e R) - H_0^{(1)}(k_i R)\right] \quad (11)$$

$$\frac{\partial}{\partial n}\left[\rho G_e(r,\alpha;r',-\alpha) - G_i(r,\alpha;r',-\alpha)\right] = \frac{\sin 2\alpha}{2i}\frac{r'}{R}\left[k_i H_1^{(1)}(k_i R) - \rho k_e H_1^{(1)}(k_e R)\right] \quad (12)$$

$$\begin{aligned} \frac{\partial^2}{\partial n \partial n'}\left[G_e(r,\alpha;r',\alpha) - G_i(r,\alpha;r',\alpha)\right] &= \frac{\left[k_e H_1^{(1)}(k_e|r-r'|) - k_i H_1^{(1)}(k_i|r-r'|)\right]}{2i|r-r'|} \\ &\xrightarrow[r\to r']{} \frac{1}{2\pi}\left(k_e^2 \ln\frac{k_e|r-r'|}{2} - k_i^2\ln\frac{k_i|r-r'|}{2}\right) \end{aligned} \quad (13)$$

$$\begin{aligned} &\frac{\partial^2}{\partial n \partial n'}\left[G_e(r,\alpha;r',-\alpha) - G_i(r,\alpha;r',-\alpha)\right] \\ &\quad = \frac{1}{2iR}\left\{\left[\cos 2\alpha - \frac{2rr'\sin^2 2\alpha}{R^2}\right]\left[k_i H_1^{(1)}(k_i R) - k_e H_1^{(1)}(k_e R)\right]\right. \\ &\qquad \left. + \frac{rr'\sin^2 2\alpha}{R}\left[k_i^2 H_0^{(1)}(k_i R) - k_e^2 H_0^{(1)}(k_e R)\right]\right\}. \end{aligned} \quad (14)$$

The distance

$$R = (r^2 + r'^2 - 2rr'\cos 2\alpha)^{1/2} \quad (15)$$

between points on opposing wedge faces is zero only as both points coalesce at the apex ($r = r' = 0$). Excitation from the line source of unit strength at (r_0, ϕ_0) gives the forcing terms

$$\begin{aligned} v_{\pm}^{\mathrm{inc}}(r) = \frac{i}{4}&\left[H_0^{(1)}\left(k_e\sqrt{r^2 + r_0^2 - 2rr_0\cos(\phi_0 - \alpha)}\right)\right. \\ &\left. \pm H_0^{(1)}\left(k_e\sqrt{r^2 + r_0^2 - 2rr_0\cos(\phi_0 + \alpha)}\right)\right] \end{aligned} \quad (16)$$

$$w_{\pm}^{\mathrm{inc}}(r) = \frac{ik_e r_0}{4}\left[\sin(\phi_0-\alpha)\frac{H_1^{(1)}\left(k_e\sqrt{r^2+r_0^2-2rr_0\cos(\phi_0-\alpha)}\right)}{\sqrt{r^2+r_0^2-2rr_0\cos(\phi_0-\alpha)}}\right.$$

$$\left.\pm\sin(\phi_0+\alpha)\frac{H_1^{(1)}\left(k_e\sqrt{r^2+r_0^2-2rr_0\cos(\phi_0+\alpha)}\right)}{\sqrt{r^2+r_0^2-2rr_0\cos(\phi_0+\alpha)}}\right]. \quad (17)$$

II. Piecewise Constant Approximation with Collocation

A moment-method expansion in terms of the pulse functions of width Δ in r,

$$P_m(r) = \begin{cases} 1, & (m-1)\Delta \le r \le m\Delta \\ 0, & \text{otherwise} \end{cases} \quad (18)$$

proceeds formally, without regard to truncation or anticipation of asymptotic behavior, as

$$v_{\pm}(r) = \sum_{m=1}^{\infty} v_m^{\pm} P_m(r), \qquad \frac{w_{\pm}(r)}{k_e} = \sum_{m=1}^{\infty} w_m^{\pm} P_m(r). \quad (19)$$

Collocation at the center points of each pulse

$$r_l = (l-1/2)\Delta \quad (20)$$

yields the coupled matrix equations

$$(1+\rho)[I][v^{\pm}] \pm [A][v^{\pm}] - [B^{\pm}][w^{\pm}] = [f^{\pm}] \quad (21)$$

$$(1+\rho)[I][w^{\pm}] \mp [C][w^{\pm}] + \rho[D^{\pm}][v^{\pm}] = \rho[g^{\pm}]. \quad (22)$$

Upon discretization, the attractive feature of integral equations of the second kind is maintained in the form of diagonal dominance. The elements of the column vectors $[f^{\pm}]$ and $[g^{\pm}]$ are the right hand sides of (7) and (8), respectively, evaluated at the collocation points

$$f_l^{\pm} = 2v_{\pm}^{\mathrm{inc}}(r_l), \qquad g_l^{\pm} = \frac{2}{k_e} w_{\pm}^{\mathrm{inc}}(r_l), \quad (23)$$

and $[I]$ is the identity matrix.

Integral forms for the dimensionless coefficients are

$$A_{lm} = \int_{(m-1)\Delta}^{m\Delta} dr' \frac{\partial}{\partial n'}\left[G_e(r_l,\alpha;r',-\alpha) - \rho G_i(r_l,\alpha;r',-\alpha)\right] \quad (24)$$

$$B^{\pm}_{lm} = k_e \int\limits_{(m-1)\Delta}^{m\Delta} dr' \left\{ \left[G_e(r_l, \alpha; r', \alpha) - G_i(r_l, \alpha; r', \alpha) \right] \right.$$

$$\left. \pm \left[G_e(r_l, \alpha; r', -\alpha) - G_i(r_l, \alpha; r', -\alpha) \right] \right\} \quad (25)$$

$$C_{lm} = \int\limits_{(m-1)\Delta}^{m\Delta} dr' \frac{\partial}{\partial n} \left[\rho G_e(r_l, \alpha; r', -\alpha) - G_i(r_l, \alpha; r', -\alpha) \right] \quad (26)$$

$$D^{\pm}_{lm} = \frac{1}{k_e} \int\limits_{(m-1)\Delta}^{m\Delta} dr' \frac{\partial^2}{\partial n \partial n'} \left\{ \left[G_e(r_l, \alpha; r', \alpha) - G_i(r_l, \alpha; r', \alpha) \right] \right.$$

$$\left. \pm \left[G_e(r_l, \alpha; r', -\alpha) - G_i(r_l, \alpha; r', -\alpha) \right] \right\}. \quad (27)$$

III. Computational Details

Machine computation is facilitated by introducing the normalized parameters

$$x' = k_e r' \qquad x_l = k_e r_l \qquad \sigma = k_i / k_e$$

$$\delta = k_e \Delta \qquad F = k_e R = (x_l^2 + x'^2 - 2 x_l x' \cos 2\alpha)^{1/2}. \quad (28)$$

The required matrix elements are now obtained by quadrature, with careful attention to extract the removable and logarithmic singularities that occur in two species of the diagonal terms ($l = m$):

$$A_{lm} = \frac{x_l \sin 2\alpha}{2i} \int\limits_{(m-1)\delta}^{m\delta} dx' \frac{1}{F} \left[\rho \sigma H_1^{(1)}(\sigma F) - H_1^{(1)}(F) \right] \quad (29)$$

$$B^{\pm}_{lm} = B^{(1)}_{lm} \pm B^{(2)}_{lm}$$

$$B^{(1)}_{lm} = \int\limits_{(m-1)\delta}^{m\delta} dx' \underbrace{\frac{\left[H_0^{(1)}(|x_l - x'|) - H_0^{(1)}(\sigma |x_l - x'|) \right]}{2i}}_{\longrightarrow -\frac{1}{\pi} \ln \sigma \text{ as } x' \to x_l} \quad (30)$$

$$B^{(2)}_{lm} = \frac{1}{2i} \int\limits_{(m-1)\delta}^{m\delta} dx' \left[H_0^{(1)}(F) - H_0^{(1)}(\sigma F) \right] \quad (31)$$

$$C_{lm} = \frac{\sin 2\alpha}{2i} \int\limits_{(m-1)\delta}^{m\delta} dx' \frac{x'}{F} \left[\sigma H_1^{(1)}(\sigma F) - \rho H_1^{(1)}(F) \right] \quad (32)$$

$$D_{lm}^{\pm} = D_{lm}^{(1)} \pm D_{lm}^{(2)}$$

$$D_{lm}^{(1)} = \int_{(m-1)\delta}^{m\delta} dx' \, \underbrace{\frac{H_1^{(1)}(|x_l - x'|) - \sigma H_1^{(1)}(\sigma|x_l - x'|)}{2i|x_l - x'|}}_{\longrightarrow \frac{1}{2\pi}\left[\ln \frac{|x_l - x'|}{2} - \sigma^2 \ln \frac{\sigma|x_l - x'|}{2}\right] \text{ as } x' \to x_l} \tag{33}$$

$$D_{lm}^{(2)} = \frac{1}{2i} \int_{(m-1)\delta}^{m\delta} dx' \, \frac{1}{F} \left\{ \left[\cos 2\alpha - \frac{2x_l x' \sin^2 2\alpha}{F^2}\right] \left[\sigma H_1^{(1)}(\sigma F) - H_1^{(1)}(F)\right] + \frac{x_l x' \sin^2 2\alpha}{F} \left[\sigma^2 H_0^{(1)}(\sigma F) - H_0^{(1)}(F)\right] \right\}. \tag{34}$$

A Fortran program that implements the above analysis is included in Appendix A. A sharp truncation of both the expansions (19) and the collocation points (20) at $l, m = N$ results in close (not large r) behavior of $v(r)$ and $w(r)$ that is surprisingly persistent. However, an accurate solution must account for the infinite domain. A physically-based approach is to include both cylindrical waves that Davis[1] extracts from an asymptotic estimate:

$$v_{\pm}(r) = \sum_{m=1}^{N} v_m^{\pm} P_m(r) + v_{N+1}^{\pm} \frac{e^{ik_i r}}{\sqrt{k_i r}} + v_{N+2}^{\pm} \frac{e^{ik_e r}}{\sqrt{k_e r}} \tag{35}$$

$$\frac{w_{\pm}(r)}{k_e} = \sum_{m=1}^{N} w_m^{\pm} P_m(r) + w_{N+1}^{\pm} \frac{e^{ik_i r}}{\sqrt{k_i r}} + w_{N+2}^{\pm} \frac{e^{ik_e r}}{\sqrt{k_e r}}. \tag{36}$$

The infinite-range, highly oscillatory integrals required by this modification are calculated with the aid of a $\pi/2$ rotation in the complex plane, resulting in exponentially decaying integrands that are more suitable for direct quadrature.

IV. Extensions and Conclusions

Accurate and efficient implementation of the above mathematics requires careful numerical analysis. The proposed asymptotic ansatz to account for the far behavior of the unknown surface distributions is best verified first for a simpler, known problem: the soft (or hard) half-plane. Although the selected set of coupled integral equations does introduce a *pair* of unknown functions, as opposed to some methods that require only a single unknown, the attractive features are weakly singular kernels and second-kind structure. Also, having direct access to both the scalar field and its normal derivative on the boundary is a benefit to the techniques of far-field evaluation based on various Green's functions.

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Appendix A. Fortran Program for Coupled Integral Equations

```
C  FILE "COUPSEQ FORTRAN A"  -  COUPLED INTEGRAL EQUATIONS FOR
C  DIELECTRIC WEDGE SCATTERING.
C  15 MAY 1994   R.W.SCHARSTEIN   UNIVERSITY OF ALABAMA  EE DEPT.
C
      PARAMETER (NN=50)
      EXTERNAL FA,FAREAL,FAIMAG,FB1,FB1REAL,FB1IMAG,FB2,FB2REAL,FB2IMAG
      EXTERNAL FC,FCREAL,FCIMAG,FD2,FD2REAL,FD2IMAG
      EXTERNAL FD1A,FD1AREA,FD1AIMA,FD1B,FD1BREA,FD1BIMA
      COMMON /COM/XL,ALPHA,RHO,SIGMA
      COMMON /SOURCE/X0,PHI0,ALPHA1
      COMPLEX A(NN,NN),B1(NN,NN),B2(NN,NN),C(NN,NN),D1(NN,NN),D2(NN,NN)
      COMPLEX B(NN,NN),D(NN,NN)
      COMPLEX Q(2*NN,2*NN),Y(2*NN),VINC,WINC,V,W
      COMPLEX CROUT(2*NN,4*NN),TEMP,DET,BIG,ERROR
      DIMENSION LP(2*NN)
      INTEGER P
C
      DATA PI/3.1415927/
C
C  CHOOSE EVEN (P=2) OR ODD (P=1) SYMMETRY
      P=2
C
C  QUADRATURE PARAMETERS
      EREL = 0.01
      EABS = 0.
C  PHYSICAL CONSTANTS
      ALPHA = PI/6.
      ALPHA1 = ALPHA
      RHO = 0.1
      SIGMA = SQRT(10.)
C  SOURCE COORDINATES
      X0 = 2.
      PHI0 = PI/2.
C  MORE INTEGRATION PARAMETERS
      XPMAX = 25.
      DELTA = XPMAX/FLOAT(NN)
C
      DO 10  L=1,NN
      XL = (FLOAT(L)-0.5)*DELTA
      DO 10  M=1,NN
      X1 = FLOAT(M-1)*DELTA
      X2 = FLOAT(M)*DELTA
      CALL QDAG(FAREAL,X1,X2,EABS,EREL,6,AR,EREST)
      CALL QDAG(FAIMAG,X1,X2,EABS,EREL,6,AI,EREST)
      A(L,M) = CMPLX(AR,AI)*XL*SIN(2.*ALPHA)/CMPLX(0.,2.)
C     WRITE(6,100) L,M,A(L,M)
      CALL QDAG(FB1REAL,X1,X2,EABS,EREL,6,B1R,EREST)
      CALL QDAG(FB1IMAG,X1,X2,EABS,EREL,6,B1I,EREST)
      B1(L,M) = CMPLX(B1R,B1I)
C     WRITE(6,100) L,M,B1(L,M)
      CALL QDAG(FB2REAL,X1,X2,EABS,EREL,6,B2R,EREST)
      CALL QDAG(FB2IMAG,X1,X2,EABS,EREL,6,B2I,EREST)
      B2(L,M) = CMPLX(B2R,B2I)/CMPLX(0.,2.)
C     WRITE(6,100) L,M,B2(L,M)
      CALL QDAG(FCREAL,X1,X2,EABS,EREL,6,CR,EREST)
      CALL QDAG(FCIMAG,X1,X2,EABS,EREL,6,CI,EREST)
      C(L,M) = CMPLX(CR,CI)*SIN(2.*ALPHA)/CMPLX(0.,2.)
C     WRITE(6,100) L,M,C(L,M)
      IF(L.EQ.M) GO TO 20
      CALL QDAG(FD1AREA,X1,X2,EABS,EREL,6,D1R,EREST)
```

```
      CALL QDAG(FD1AIMA,X1,X2,EABS,EREL,6,D1I,EREST)
      D1(L,M) = CMPLX(D1R,D1I)
      GO TO 30
   20 CONTINUE
      CALL QDAG(FD1BREA,X1,X2,EABS,EREL,6,D1R,EREST)
      CALL QDAG(FD1BIMA,X1,X2,EABS,EREL,6,D1I,EREST)
      D1(L,M) = CMPLX(D1R,D1I) + DELTA*((1.-SIGMA**2)*(ALOG(DELTA/4.)
     &  -1.) - SIGMA**2*ALOG(SIGMA))/6.2831853
   30 CONTINUE
C     WRITE(6,100) L,M,D1(L,M)
      CALL QDAG(FD2REAL,X1,X2,EABS,EREL,6,D2R,EREST)
      CALL QDAG(FD2IMAG,X1,X2,EABS,EREL,6,D2I,EREST)
      D2(L,M) = CMPLX(D2R,D2I)/CMPLX(0.,2.)
C     WRITE(6,100) L,M,D2(L,M)
   10 CONTINUE
C
C  BOTH EVEN (+) AND ODD (-) SYMMETRY PROBLEMS
      DO 40  L=1,NN
      DO 40  M=1,NN
      B(L,M) = B1(L,M) + (-1)**P*B2(L,M)
   40 D(L,M) = D1(L,M) + (-1)**P*D2(L,M)
C  THE BIG SUPER-MATRIX Q
      DO 42  L=1,NN
      DO 42  M=1,NN
      Q(L,M) = (-1)**P*A(L,M)
      IF(L.EQ.M) Q(L,M) = Q(L,M) + 1.+RHO
      Q(L,M+NN) = -B(L,M)
      Q(L+NN,M) = RHO*D(L,M)
      Q(L+NN,M+NN) = (-1)**(P+1)*C(L,M)
      IF(L.EQ.M) Q(L+NN,M+NN) = Q(L+NN,M+NN) + 1.+RHO
   42 CONTINUE
C  THE BIG RIGHT-HAND COLUMN VECTOR
      DO 44  L=1,NN
      XL = (FLOAT(L)-0.5)*DELTA
      Y(L) = 2.*VINC(XL,P)
   44 Y(L+NN) = 2.*RHO*WINC(XL,P)
C  INVERT THE MATRIX Q AND SOLVE THE SYSTEM OF LINEAR EQUATIONS
C  USING CROUTC (INCLUDED)
      DO 46  L=1,2*NN
      DO 46  M=1,2*NN
   46 CROUT(L,M) = Q(L,M)
      CALL CROUTC(2*NN,2*NN,0,CROUT,1.E-7,DET,IERR,LP)
      WRITE(6,700) IERR
  700 FORMAT(//,7X,'SUBROUTINE "CROUTC"  IERR = ',I3)
C  CHECK MATRIX INVERSION
      BIG = CMPLX(0.,0.)
      DO 57  K=1,2*NN
      DO 57  L=1,2*NN
      TEMP = CMPLX(0.,0.)
      DO 58  M=1,2*NN
   58 TEMP = TEMP + CROUT(K,M+2*NN)*Q(M,L)
      ERROR = TEMP
      IF(K.EQ.L) ERROR = TEMP - CMPLX(1.,0.)
      IF(CABS(ERROR).GT.CABS(BIG)) BIG = ERROR
   57 CONTINUE
      WRITE(6,710) BIG
  710 FORMAT(/,7X,'BIGGEST ERROR IN MATRIX INVERSION = ',2(E12.5,2X))
C  RESULTANT COLUMN VECTORS
      WRITE(6,200)
  200 FORMAT(//,4X,'N',14X,'V',22X,'W',/)
```

```
  210 FORMAT(2X,I3,2(2X,E12.5,2X,F7.2))
      DO 60 L=1,NN
      V = CMPLX(0.,0.)
      W = CMPLX(0.,0.)
      DO 62 M=1,2*NN
      V = V + CROUT(L,M+2*NN)*Y(M)
   62 W = W + CROUT(L+NN,M+2*NN)*Y(M)
      WRITE(6,210) L,CABS(V),CDEG(V),CABS(W),CDEG(W)
   60 CONTINUE
  100 FORMAT(2X,I3,2X,I3,2X,E14.7,2X,E14.7)
      STOP
      END
C
C
C
      FUNCTION FA(XP)
      COMMON /COM/XL,ALPHA,RHO,SIGMA
      COMPLEX FA,H1,H2
      F = SQRT(XL**2+XP**2-2.*XL*XP*COS(2.*ALPHA))
      H1 = CMPLX(BSJ1(SIGMA*F),BSY1(SIGMA*F))
      H2 = CMPLX(BSJ1(F),BSY1(F))
      FA = (RHO*SIGMA*H1-H2)/F
      RETURN
      END
C
C
      FUNCTION FAREAL(XP)
      COMPLEX FA
      FAREAL = REAL(FA(XP))
      RETURN
      END
C
C
      FUNCTION FAIMAG(XP)
      COMPLEX FA
      FAIMAG = AIMAG(FA(XP))
      RETURN
      END
C
C
C
      FUNCTION FB1(XP)
      COMMON /COM/ XL,ALPHA,RHO,SIGMA
      COMPLEX FB1,H1,H2
      D = ABS(XL-XP)
      IF(D.LT.0.001) GO TO 10
      H1 = CMPLX(BSJ0(D),BSY0(D))
      H2 = CMPLX(BSJ0(SIGMA*D),BSY0(SIGMA*D))
      FB1 = (H1-H2)/CMPLX(0.,2.)
      RETURN
   10 FB1 = -ALOG(SIGMA)/3.1415927
      RETURN
      END
C
C
      FUNCTION FB1REAL(XP)
      COMPLEX FB1
      FB1REAL = REAL(FB1(XP))
      RETURN
      END
```

```
C
C
      FUNCTION FB1IMAG(XP)
      COMPLEX FB1
      FB1IMAG = AIMAG(FB1(XP))
      RETURN
      END
C
C
C
      FUNCTION FB2(XP)
      COMMON /COM/ XL,ALPHA,RHO,SIGMA
      COMPLEX FB2,H1,H2
      F = SQRT(XL**2+XP**2-2.*XL*XP*COS(2.*ALPHA))
      H1 = CMPLX(BSJ0(F),BSY0(F))
      H2 = CMPLX(BSJ0(SIGMA*F),BSY0(SIGMA*F))
      FB2 = H1-H2
      RETURN
      END
C
C
      FUNCTION FB2REAL(XP)
      COMPLEX FB2
      FB2REAL = REAL(FB2(XP))
      RETURN
      END
C
C
      FUNCTION FB2IMAG(XP)
      COMPLEX FB2
      FB2IMAG = AIMAG(FB2(XP))
      RETURN
      END
C
C
C
      FUNCTION FC(XP)
      COMMON /COM/ XL,ALPHA,RHO,SIGMA
      COMPLEX FC,H1,H2
      F = SQRT(XL**2+XP**2-2.*XL*XP*COS(2.*ALPHA))
      H1 = CMPLX(BSJ1(SIGMA*F),BSY1(SIGMA*F))
      H2 = CMPLX(BSJ1(F),BSY1(F))
      FC = XP*(SIGMA*H1-RHO*H2)/F
      RETURN
      END
C
C
      FUNCTION FCREAL(XP)
      COMPLEX FC
      FCREAL = REAL(FC(XP))
      RETURN
      END
C
C
      FUNCTION FCIMAG(XP)
      COMPLEX FC
      FCIMAG = AIMAG(FC(XP))
      RETURN
      END
C
```

```
C
C
      FUNCTION FD1A(XP)
C  USE THIS ONE FOR OFF-DIAGONAL (L NOT= M) TERMS (NO SINGULARITY!)
      COMMON /COM/ XL,ALPHA,RHO,SIGMA
      COMPLEX FD1A,H1,H2
      D = ABS(XL-XP)
      H1 = CMPLX(BSJ1(D),BSY1(D))
      H2 = CMPLX(BSJ1(SIGMA*D),BSY1(SIGMA*D))
      FD1A = (H1-SIGMA*H2)/CMPLX(0.,2.*D)
      RETURN
      END
C
C
      FUNCTION FD1AREA(XP)
      COMPLEX FD1A
      FD1AREA = REAL(FD1A(XP))
      RETURN
      END
C
C
      FUNCTION FD1AIMA(XP)
      COMPLEX FD1A
      FD1AIMA = AIMAG(FD1A(XP))
      RETURN
      END
C
C
C
      FUNCTION FD1B(XP)
C  USE THIS ONE FOR DIAGONAL (L = M) TERMS (WITH SINGULARITY!)
      COMMON /COM/ XL,ALPHA,RHO,SIGMA
      COMPLEX FD1B,H1,H2
      D = ABS(XL-XP)
      IF(D.LT.0.001) GO TO 10
      H1 = CMPLX(BSJ1(D),BSY1(D))
      H2 = CMPLX(BSJ1(SIGMA*D),BSY1(SIGMA*D))
      FD1B = (H1-SIGMA*H2)/CMPLX(0.,2.*D)
      FD1B = FD1B - (ALOG(D/2.)-SIGMA**2*ALOG(SIGMA*D/2.))/6.2831853
      RETURN
   10 FD1B = CMPLX(0.,0.)
      RETURN
      END
C
C
      FUNCTION FD1BREA(XP)
      COMPLEX FD1B
      FD1BREA = REAL(FD1B(XP))
      RETURN
      END
C
C
      FUNCTION FD1BIMA(XP)
      COMPLEX FD1B
      FD1BIMA = AIMAG(FD1B(XP))
      RETURN
      END
C
C
C
```

```
      FUNCTION FD2(XP)
      COMMON /COM/XL,ALPHA,RHO,SIGMA
      COMPLEX FD2,H1,H2,H3,H4
      F = SQRT(XL**2+XP**2-2.*XL*XP*COS(2.*ALPHA))
      H1 = CMPLX(BSJ1(SIGMA*F),BSY1(SIGMA*F))
      H2 = CMPLX(BSJ1(F),BSY1(F))
      H3 = CMPLX(BSJ0(SIGMA*F),BSY0(SIGMA*F))
      H4 = CMPLX(BSJ0(F),BSY0(F))
      Z = XL*XP*(SIN(2.*ALPHA))**2
      FD2 = (COS(2.*ALPHA)-2.*Z/F**2)*(SIGMA*H1-H2)+Z*(SIGMA**2*H3-H4)/F
      FD2 = FD2/F
      RETURN
      END
C
C
      FUNCTION FD2REAL(XP)
      COMPLEX FD2
      FD2REAL = REAL(FD2(XP))
      RETURN
      END
C
C
      FUNCTION FD2IMAG(XP)
      COMPLEX FD2
      FD2IMAG = AIMAG(FD2(XP))
      RETURN
      END
C
C
C
      FUNCTION VINC(XP,P)
      COMMON /SOURCE/ X0,PHI0,ALPHA
      COMPLEX VINC,H1,H2
      INTEGER P
      G1 = SQRT(XP**2+X0**2-2.*XP*X0*COS(PHI0-ALPHA))
      G2 = SQRT(XP**2+X0**2-2.*XP*X0*COS(PHI0+ALPHA))
      H1 = CMPLX(BSJ0(G1),BSY0(G1))
      H2 = CMPLX(BSJ0(G2),BSY0(G2))
      VINC = (H1+(-1)**P*H2)*CMPLX(0.,1.)/4.
      RETURN
      END
C
C
C
      FUNCTION WINC(XP,P)
      COMMON /SOURCE/ X0,PHI0,ALPHA
      COMPLEX WINC,H1,H2
      INTEGER P
      G1 = SQRT(XP**2+X0**2-2.*XP*X0*COS(PHI0-ALPHA))
      G2 = SQRT(XP**2+X0**2-2.*XP*X0*COS(PHI0+ALPHA))
      H1 = CMPLX(BSJ1(G1),BSY1(G1))
      H2 = CMPLX(BSJ1(G2),BSY1(G2))
      WINC = (SIN(PHI0-ALPHA)*H1/G1+(-1)**P*SIN(PHI0+ALPHA)*H2/G2)
     & *CMPLX(0.,X0)/4.
      RETURN
      END
C
C
C
      FUNCTION CDEG(Z)
```

```
C
C  PHASE ANGLE IN DEGREES OF A COMPLEX NUMBER Z.
C
      COMPLEX Z
      ZI = AIMAG(Z)
      ZR = REAL(Z)
      IF((ZI.EQ.0.).AND.(ZR.EQ.0.)) GO TO 10
      CDEG = ATAN2(ZI,ZR)*57.29578
      RETURN
   10 CDEG = 0.
      RETURN
      END
C
C
C
      SUBROUTINE CROUTC(MR,NR,NCC,A,ZMCH,DT,IERR,LP)

C     CROUT (1) OPERATES ON A COEFFICIENT MATRIX TO SOLVE A SYSTEM OF
C     SIMULTANEOUS EQUATIONS OR TO COMPUTE AN INVERSE AND (2)
C     COMPUTES A DETERMINANT.

C     CROUT REDUCES THE ORIGINAL MATRIX AND RIGHT-HAND SIDES UNTIL
C     UPON COMPLETION THE REDUCED MATRIX REPLACES THE ORIGINAL
C     MATRIX AND THE SOLUTIONS REPLACE THE RIGHT-HAND SIDES.

      COMPLEX * 8 A,DT,TEMPC

      DIMENSION A(1), LP(1)
      IF(NR.GT.MR) GO TO 210
      MTX = MR*NR
      MRA = MR + 1
      MRS = MR - 1
      MDN = MR - NR
      MTR = MTX - MDN
      MTRA = MTX + 1
      DT = (1.,0.)
      IERR = 0
      NRS = NR - 1
      DO 2 I = 1,NR
      LP(I) = I
    2 CONTINUE
      IF(NCC)210,805,1001
  805 NTC = NR + NR
      MTT = MR*NTC - MDN
      J = MTRA
      DO 19 K = MTRA,MTT,MR
      JF = K + NRS
      DO 18 KX = K,JF
      A(KX) = (0.,0.)
   18 CONTINUE
      A(J) = (1.,0.)
      J = J + MRA
   19 CONTINUE
      GO TO 1
 1001 NTC = NR + NCC
      MTT = MR*NTC - MDN
    1 IF(NTC.LE.NR) GO TO 210
      DO 70 I = 1,NR
      IS = I - 1
      II = MR*IS + I
```

```
      IISB = II - 1
      IIAD = II + MR
      ICF = MR*I - MDN
      ICS = ICF - NRS
      IIA = II + 1
      TEMP = 0.
      DO 31 J = II,MTT,MR
      IF(I.EQ.1) GO TO 33
      KF = J - 1
      KS = J - I + 1
      KX = I
      DO 30 K = KS,KF
      A(J) = A(J) - A(KX)* A(K)
      KX = KX + MR
30    CONTINUE
33    IF(J.GT.MTR) GO TO 31
      IF(CABS(A(J)).LE.TEMP) GO TO 31
      TEMP = CABS(A(J))
      NX = J/MR + 1
31    CONTINUE
      IF(I.EQ.NR) GO TO 35
      IF(NX.EQ.I) GO TO 35
      ITEMP = LP(NX)
      LP(NX) = LP(I)
      LP(I) = ITEMP
      LPIS = MR*NX - MRS
      DO 34 K = ICS,ICF
      TEMPC = A(K)
      A(K) = A(LPIS)
      A(LPIS) = TEMPC
      LPIS = LPIS + 1
34    CONTINUE
      DT = - DT
35    DT = DT * A(II)
      IF(ZMCH - CABS(A(II))) 45,45,220
45    DO 46 J = IIAD,MTT,MR
      A(J) = A(J)/A(II)
46    CONTINUE
      IF(I.EQ.1) GO TO 70
      IF(I.EQ.NR) GO TO 78
      DO 47 M = IIA,ICF
      KX = M - ICS + 1
      DO 48 KY = ICS,IISB
      A(M) = A(M) - A(KX) * A(KY)
      KX = KX + MR
48    CONTINUE
47    CONTINUE
70    CONTINUE
78    NRTAD = MTX + NR
      DO 180 I = 1,NRS
      IREV = NRTAD - I
      KRS = IREV - MR*I
      DO 170 IRCNT = IREV,MTT,MR
      KCS = IRCNT + 1
      DO 160 K = KRS,MTR,MR
      A(IRCNT) = A(IRCNT) - A(KCS) * A(K)
      KCS = KCS + 1
160   CONTINUE
170   CONTINUE
180   CONTINUE
```

```
      DO 6 I = 1,NRS
9     IF(LP(I).EQ.I) GO TO 6
      NX = LP(I)
      LP (I) = LP(NX)
      LP(NX) = NX
      IXS = MTX + I
      IY = MTX + NX
      DO 7 IX = IXS,MTT,MR
      TEMPC = A(IX)
      A(IX) = A(IY)
      A(IY) = TEMPC
      IY = IY + MR
7     CONTINUE
      GO TO 9
6     CONTINUE
      RETURN
210   IERR = 2
      NTC = NR
      DT = (999999999.,999999999.)
      RETURN
220   IERR = 1
      RETURN
      END
```

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Robert W. Scharstein and Anthony M.J. Davis

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Two complementary analyses of the time-harmonic scattering by a penetrable wedge are presented. The distance from the apex (appropriately scaled by the wavenumber in the exterior region) of the exciting line source is the single length scale in this infinite-domain boundary value problem. The work summarized herein represents two mathematical approaches (among a series of candidates) to solve this important scattering problem and to visualize the wave physics.

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